SAINT IGNATIUS' COLLEGE

Trial Higher School Certificate

2008

MATHEMATICS EXTENSION 1

8:45am – 10:50am
Friday 5th September

Directions to Students

• Reading Time: 5 minutes	• Total Marks: 84
• Working Time: 2 hours	
• Write using blue or black pen (sketches in pencil).	• Attempt Questions 1 – 7
• Board approved calculators may be used	• All questions are of equal value
• A table of standard integrals is provided at the back of this paper.	
• All necessary working should be shown in every question.	
• **Answer each question in the booklets provided and clearly label your name and teacher's name.**	

QUESTION 1 (use a SEPARATE writing booklet)

(a) If $x = \dfrac{nx_1 + mx_2}{m+n}$ and $x_1 = -2$ *and* $x_2 = 3$, find x when

(i) $m = 3$, $n = 2$ (1M)

(ii) $m = 3$, $n = -2$ (1M)

(b) Noting that $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$, write down, without explanation, the value of:

(i) $\lim\limits_{x \to 0} \dfrac{\sin 2008x}{x}$ (1M)

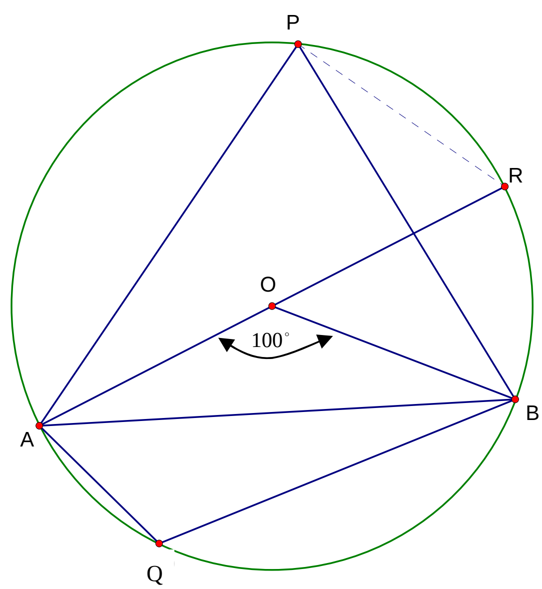
(ii) $\lim\limits_{x \to 0} \dfrac{\sin(\pi - 2x)}{2008x}$, assuming that $\sin(\pi - 2x) = \sin 2x$. (1M)

(c) Write down the derivative, with respect to x, of

(i) $\cos^{-1} x$ (1M)

(ii) e^{7x} (1M)

(d) In the diagram, P,R,B,Q and A are points on the circumference of a circle centre O and AOR is a diameter.



(i) Given $A\hat{O}B = 100^\circ$, briefly explain why $A\hat{P}B = 50^\circ$ (1M)

(ii) Write down, without explanation, the size of

(a) $A\hat{Q}B$ (1M)

(b) $B\hat{P}R$ (1M)

(e) Noting that $\frac{d}{dx}\left[e^x \sin x\right] = e^x \sin x + e^x \cos x$ evaluate

$$\int_0^{\frac{\pi}{2}} e^x (\sin x + \cos x)dx$$ (2M)

(f) Given that if $y = 7^x$, then $\frac{dy}{dx} = A(7^x)$, where A is a constant,

write down the exact value of A. (1M)

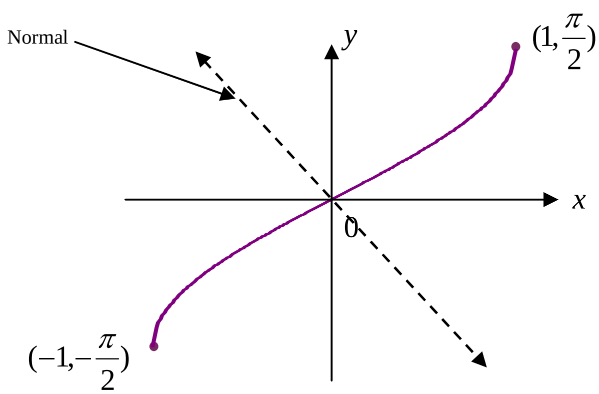
QUESTION 2 (use a SEPARATE writing booklet)

(a) Write down the fifth term in the expansion by the binomial theorem of

(i) $(a+b)^8$ (1M)

(ii) $(3x+\frac{1}{x})^8$ (2M)

(b) Focus on the graph of $y=\sin^{-1}x$ below. State (answers only are required)



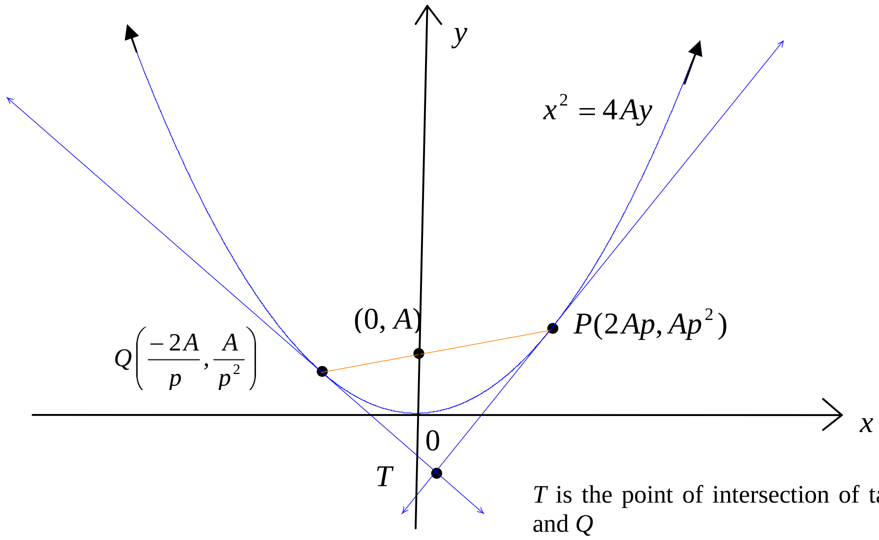
(i) The domain of *x*. (1M)

(ii) The gradient of the normal at O the origin. (1M)

(iii) The equation of the horizontal line in which the curve $y=\sin^{-1}x$ may be reflected, to obtain a sketch of $y=\cos^{-1}x$. (1M)

(c) Note that if $-\pi \le \theta \le \pi$, the solution to $\cos\theta=\frac{1}{2}$ is $\theta=\pm\frac{\pi}{3}$.
Assuming the above, write down in radians, the general solution of the equation $\cos\theta=\frac{1}{2}$. (2M)

(d)



PQ is a focal chord of the parabola

$$x^2 = 4Ay$$

The tangent at P has equation $y - px + Ap^2 = 0$..........(α).

The tangent at Q has equation $y + \frac{x}{p} + \frac{A}{p^2} = 0$............$(\beta)$.

(i) Do not prove the above but solve the equations (α) and (β) to show that the *x*- coordinate of T is $A(p - \frac{1}{p})$. (2M)

(ii) Determine the *y*- coordinate of T. (2M)

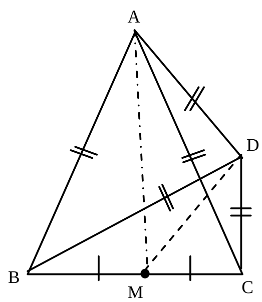
QUESTION 3 (use a SEPARATE writing booklet)

(a) Consider the polynomial $P(x) = x^3 + Ax^2 - 2008$.

If *(x-2)* is a factor of P(*x*) find A. (2M)

(b) Use the principle of mathematical induction to show that $(2008^n - 1)$ is divisible by 9 for all positive integers *n*. (4M)

(c) The figure below ABCD is a regular tetrahedron, with

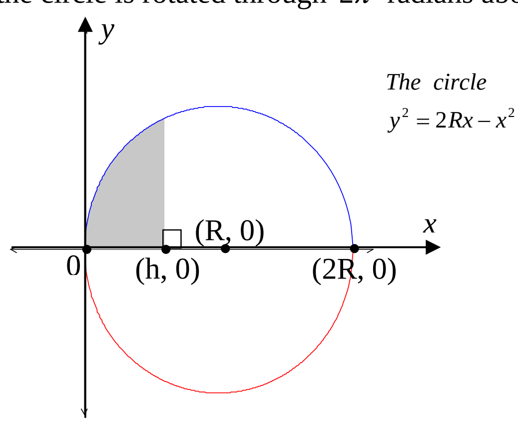


AB = AC =BC = BD = DC = AD =20*cm*.

(i) Draw the triangle BCD in your writing booklet and mark 'M' the midpoint of BC. Prove that DM = $10\sqrt{3}cm$. (2M)

(ii) Determine the size of $A\hat{M}D$. (2M)

(d) A circle centre (R, 0) and radius R has equation $y^2 = 2Rx - x^2$ and is drawn below. The shaded portion of the circle is rotated through 2π radians about the x –axis.



Prove that the volume V of the spherical cap generated is given by

$$V = \pi h^2 (R - \frac{h}{3}).$$ (2M)

x

QUESTION 4 (use a SEPARATE writing booklet)

(a) A point P moves along the parabola whose equation is $y = \frac{x^2}{8}$.

The *x*-coordinate of P increases at the constant rate of 8cm/sec.
At what rate is the *y*-coordinate increasing when $x = 4$? (2M)

(b) In solving a problem on Newton's Law of Cooling, Ignatius correctly arrives at the equation.

$$T = 20 + 980e^{-kt}$$

Where *T* is the temperature of a body at time *t*.

(i) Given that at $t = 100$, $T = 510$, show $k = \frac{\ln 2}{100}$. (1M)

(ii) Find T when $t = 200$. (1M)

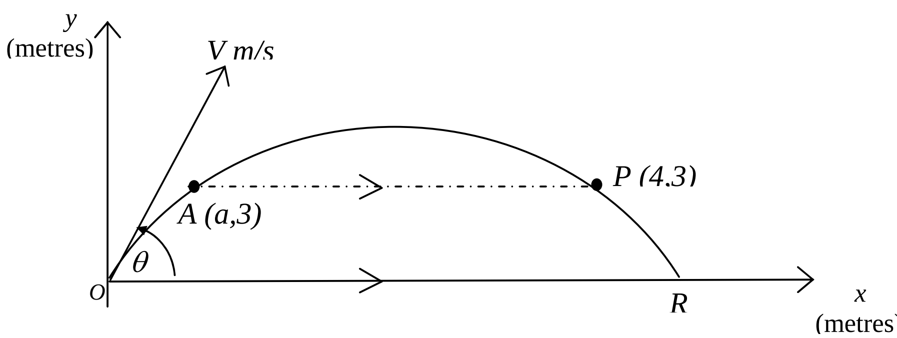
(c) If the equation $f(x) = 0$ has a root near $x = a$, it can be shown that, in general, a closer approximation to the root is a_1, where $a_1 = a - \frac{f(a)}{f'(a)}$. Do <u>not</u> prove this result, known as Newton's method of approximating roots, but draw and label:

(i) One diagram showing the regular situation in which a_1 is a better approximation than *a*. (2M)

(ii) A second diagram, showing a case in which Newton's method does <u>not</u> apply, even though $f(x)$ has a root near $x = a$. (1M)

<u>Question 4 continues on the next page.</u>

(d) A particle is projected from a point O with a speed of V *m/s* at an angle of θ to the horizontal. Air resistance is to be neglected and g ms^{-2} is the acceleration due to gravity, vertically downwards.



In solving the following problems, you may assume that the Cartesian equation of the path of the projectile is given by:

$$y = \frac{-gx^2}{2V^2}\left(1+\tan^2\theta\right) + x\tan\theta \text{ . (Do \underline{NOT} prove this result)}$$

You are given $V^2 = 8g$ and that the particle passes through a point $P(4,3)$.

(i) Prove that $\tan\theta = 2$, without using the result of (ii) below. (2M)

(ii) Assume that $\tan\theta = 2$, to show that the range $OR = 6.4\,m$. (2M)

(iii) The point $A = (a,3)$ lies on the trajectory, where AP is parallel to the x-axis. What is the value of a? (1M)

QUESTION 5 (use a SEPARATE writing booklet)

(a) Noting that $\log_e e^N = N$ show by setting $u = \log_e x$, that (3M)

$$2\int_{e^2}^{e^5} \frac{\log_e x}{x}\, dx = 21$$

(b) Find the values of x and y which satisfy the simultaneous equations

$$\cos^{-1} x + 3\sin^{-1} y = \pi \,.......(\alpha)$$
$$\cos^{-1} x - 3\sin^{-1} y = 0 \,.......(\beta)$$

(3M)

(c) Here you may use, without proof, the identity

$$\sin 3A = 3\sin A - 4\sin^3 A \,..........(\alpha)$$

(i) By direct substitution show α is true if $A = \dfrac{\pi}{6}$. (1M)

(ii) Use α to find, in exact form, the smallest positive value of x

which satisfies the equation $6x - 8x^3 = 1$. (3M)

(suggestion: let $x = \sin A$)

(d) Assuming that

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) \,........\delta$$

factorise

$$(\alpha - \beta)^3 + (\beta - \gamma)^3 + (\gamma - \alpha)^3 \,.$$ (2M)

QUESTION 6 (use a SEPARATE writing booklet)

(a) A particle, executing Simple Harmonic Motion, is moving along the x-axis with velocity v, and acceleration $\ddot{x}$.

(i) Show that $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{v^2}{2}\right)$. (1M)

(ii) Assuming that in the usual notation $\ddot{x} = -n^2 x$ and that the particle is released from rest from the point where $x = a$, and at zero time, show that:

(α) $v = -n\sqrt{a^2 - x^2}$. (2M)

(β) $x = a\cos nt$. (2M)

(b) In Aloha Harbour the height of the water above the low tide mark is given by the rule $x = 2(1 + \cos\theta)$ metres, where $\theta = \dfrac{\pi t}{390}$ radians and t is the time in minutes after high tide. (see the diagram on the next page)

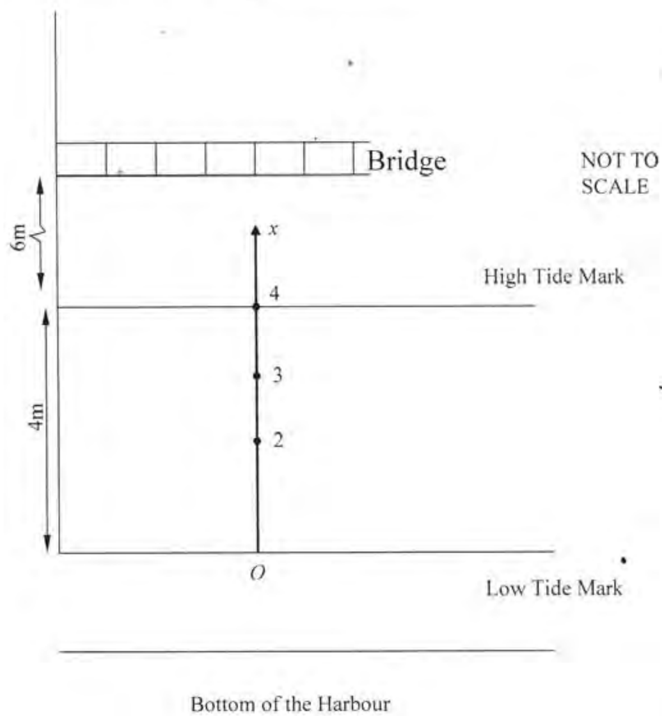
Explain why the time interval between high and low tide is 390 minutes. (2M)

(c) Given $t = 65$, show that the rate at which the water level is falling is approximately 8mm per minute. (2M)

Question 6 continues on the next page.

(d)

The diagram below may be of help to you



The Aloha Harbour Bridge is 10 metres above the low water mark. A pleasure yacht can just sail under the bridge when the distance between the bridge and the water is 7 metres. How long after high tide will it be before the yacht can first sail under the bridge? (3M)

QUESTION 7 (use a SEPARATE writing booklet)

(a) You may assume that $x^4(1-x)^4 = x^8 - 4x^7 + 6x^6 - 4x^5 + x^4$

Let $J = \int_0^1 x^4(1-x)^4\,dx$.

Show that $J = \dfrac{1}{630}$ (3M)

(b) Let $E = \dfrac{x^4(1-x)^4}{x^2+1}$.

Noting the assumption in part (a) and using the process of long division for polynomials, show that

$$E = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}$$ (3M)

(c) Let $I = \int_0^1 \dfrac{x^4(1-x)^4}{1+x^2}\,dx$. Use the result in part (b) to prove that (3M)

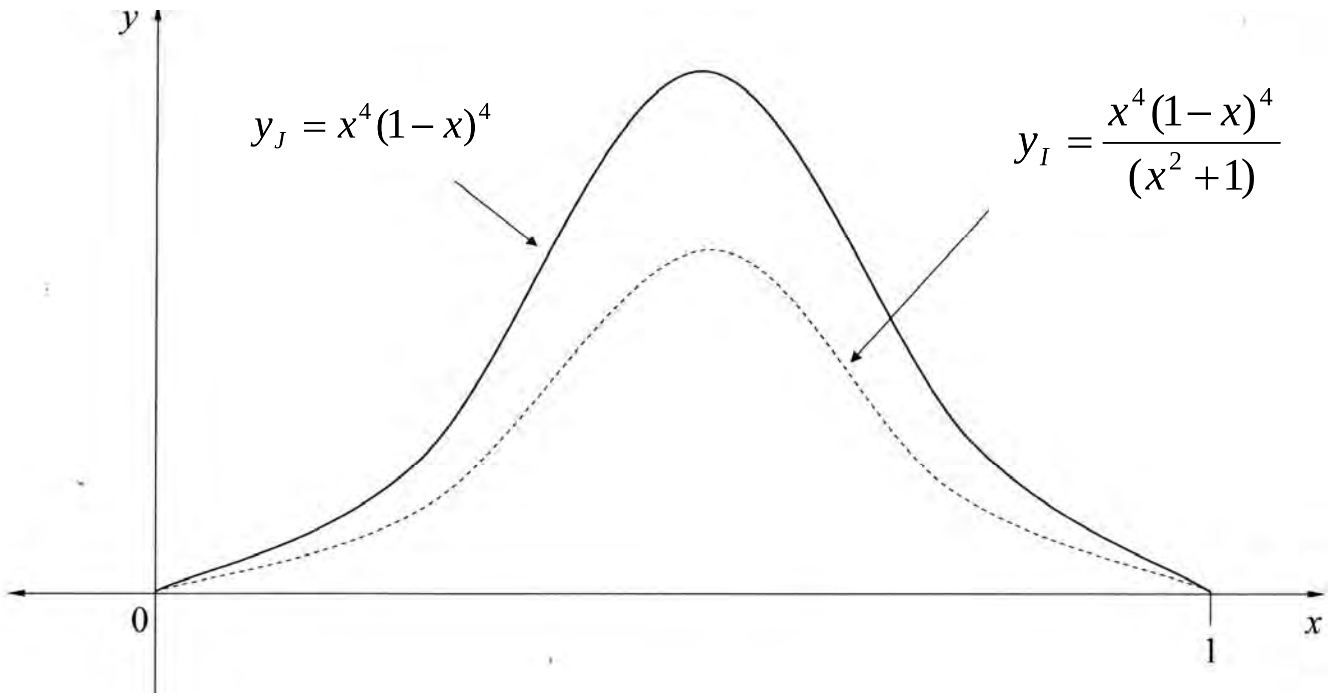
$$I = \frac{22}{7} - \pi.$$

(d) Assuming, without any explanation, that $J > I > \dfrac{J}{2}$, deduce from (a) and (c) that

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}.$$ (2M)

Question 7 continues on the next page.

(e)



Use the diagram above, or otherwise, to explain why $J > I$. (1M)

Suggested Solutions

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Solutions, Marking Scheme & Comments

QUESTION 1

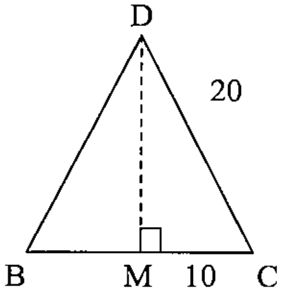
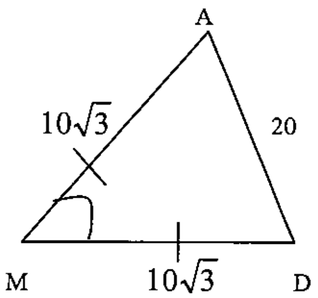
Suggested Solutions	Max Mark	Your Mark	Marker's Comments
(a) (i) $x_{(i)}=\dfrac{2\times(-2)+(3\times3)}{(3+2)}=\dfrac{5}{5}=1$	1		
(ii) $x_{(ii)}=\dfrac{(-2\times-2)+(3\times3)}{(3-2)}=\dfrac{13}{1}=13$	1		
(Award the mark if the candidate makes just one error)			
(b) (i) $E_{(i)}=(2008)\lim_{x\to0}\left(\dfrac{\sin 2008x}{2008x}\right)=2008\times1=2008$	1		
(ii) $E_{(ii)}=\lim_{x\to0}\left(\dfrac{\sin 2x}{2x}\right)\times\dfrac{1}{1004}=\dfrac{1}{1004}$	1		
(c) (i) $\dfrac{-1}{\sqrt{1-x^2}}$, $\lvert x\rvert\le1$	1		
(ii) $7e^{7x}$	1		
(d) (i) In the given circle, $A\hat{O}B$ is the angle at the centre, subtended by the arc AQB. $A\hat{P}B$ is the angle at the circumference of the same circle, subtended by the arc AQB. $\therefore A\hat{P}B=\dfrac{1}{2}A\hat{Q}B=\dfrac{100^\circ}{2}=50^\circ$.	1		
(ii) (a) 130°	1		
(b) 40°	1		
(e) $I=\left[e^x\sin x+e^x\cos x\right]_0^{\frac{\pi}{2}}$ $=\left[e^{\frac{\pi}{2}}+0\right]-[0+1]$ $=e^{\frac{\pi}{2}}-1$ *(No penalty for one error)*	2		
(f) $A=\ln7$	1		

QUESTION 2			
Suggested Solutions	**Max Mark**	**Your Mark**	**Marker's Comments**
(a) (i) $T_5 = \binom{8}{4} a^4 b^4$	1		
(ii) $T_5 = \binom{8}{4}(3x)^4\left(\frac{1}{x}\right)^4 = \frac{8.7.6.5}{4.3.2.1} 81.x^0 = 5670$ *(in part (ii) no penalty for one error)*	2		
(b) (i) $-1 \le x \le 1$	1		
(ii) -1	1		
(iii) $y = \frac{\pi}{4}$	1		
(c) General Solution is $\theta = 2\pi n \pm \frac{\pi}{3}$ where n is an integer, positive, negative or zero. *Award 1 mark for giving some new correct value of θ in radians.*	2		
(d) (i) Eliminate y to get $x\left(p+\frac{1}{p}\right) - A\left(p^2 - \frac{1}{p^2}\right) = 0$ So $x\left(p+\frac{1}{p}\right) = A\left(p^2 - \frac{1}{p^2}\right)$ $= A\left(p-\frac{1}{p}\right)\left(p+\frac{1}{p}\right)$ $\therefore x = A\left(p-\frac{1}{p}\right)$	2		
(ii) Substitute $x = A\left(p-\frac{1}{p}\right)$ into $y = px - Ap^2$ to get $y = p\left[Ap - \frac{A}{p}\right] - Ap^2 = Ap^2 - A - Ap^2 = -A$ The y coordinate of T is $-A$	2		

QUESTION 3

Suggested Solutions	Max Mark	Your Mark	Marker's Comments
(a) Let $f(x)=x^3+Ax^2-2008$ then $f(2)=0$ $\therefore 0=8+4A-2008 \therefore A=500$	2		
(b) **Step 1:** When $n=1$, $2008^1-1=2007=223\times9$, Which is divisible by 9, and Thus the statement is true for $n=1$. **Step 2:** Assume the statement true for $n=k$ i.e. assume that 2008^k-1 is divisible by 9, i.e. that $\frac{2008^k-1}{9}$ is some integer M. i.e. $\frac{2008^k-1}{9}=M$, i.e. $2008^k-1=9M \Rightarrow 2008^k=9M+1......**$ We now prove that the statement is true for $n=k+1$ i.e. $2008^{k+1}-1$ is divisible by 9 Now $2008^{k+1}-1=2008^1\cdot2008^k-1$ $=2008(9M+1)-1....**(by\ our\ assumption)$ $=9\times2008M+2008-1$ $=9\times2008M+2007$ $=9(2008M+223)$ $=9I$ (where I is a positive integer) which is divisible by 9 Thus, if 2008^k-1 is divisible by 9 when $n=k$, then it is also divisible by 9 when $n=k+1$ **Step 3:** Since the statement is true for $n=1$, it is true for $n=1+1=2$, and $n=2+1=3$, and so on for all positive integral values of n.	4		

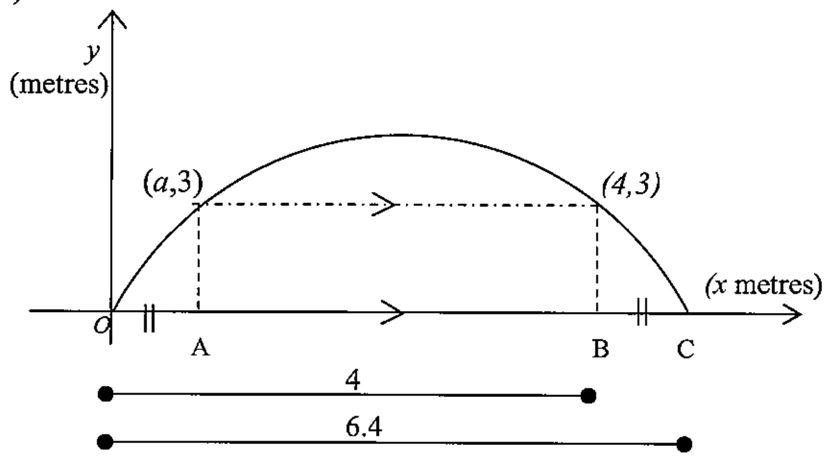
QUESTION 3

Suggested Solutions	Max Mark	Your Mark	Marker's Comments
(c) (i)  From symmetry, $D\hat{M}C = 90^\circ$ In right angle triangle DMC $DM^2 = 20^2 - 10^2 = 300$ $= 100 \times 3$ $\therefore DM = 10\sqrt{3}$ cm	2		
(c) (ii)  Apply cosine rule to ΔAMD. $\cos A\hat{M}D = \frac{300+300-400}{600} = \frac{1}{3}$ $\therefore A\hat{M}D = \cos^{-1}\left(\frac{1}{3}\right) \approx 70^\circ 32'$	2		
(d) $V = \pi\int_0^h (2Rx - x^2)\,dx$ $= \pi\left[Rx^2 - \frac{x^3}{3}\right]_0^h$ $= \pi\left[Rh^2 - \frac{h^3}{3}\right] - [0]$ $= \pi h^2\left[R - \frac{h}{3}\right]$	2		

QUESTION 4

Suggested Solutions	Max Mark	Your Mark	Marker's Comments
(a) Use $\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$ where $\frac{dy}{dx}=\frac{x}{4}$ and $\frac{dx}{dt}=8$ When $x=4$, $\frac{dy}{dt}=\left(\frac{4}{4}\times 8\right)=8$ cm/sec	2		
(b) (i) Substitute values given to get $510=20+980e^{-100K}$ $\therefore e^{-100K}=\frac{490}{980}=\frac{1}{2} \longrightarrow \therefore K=\frac{\ln 2}{100}$	1		
(ii) Here $T=20+980e^{-2\ln 2}$ $=20+245$ $=265$	1		
(c) (i) See Page 14.	2		
(ii) See Pages 15 and 16.	1		
(d) (i) Into the given equation, substitute $V^2=8g$, $x=4$, $y=3$ (data) to get $3=\frac{-16g}{2\times 8g}\left(1+\tan^2\theta\right)+4\tan\theta$ $3=-1-\tan^2\theta+4\tan\theta$ So $\tan^2\theta-4\tan\theta+4=0$ $\therefore \left(\tan\theta-2\right)^2=0$ i.e. $\tan\theta=2$ (twice) **Note.** That no other value of $\tan\theta$ satisfies.	2		
(ii) Set $y=0$, to get $0=\frac{-x^2}{16}\left(1+4\right)+2x$ $\therefore \frac{x}{16}\left(5x-32\right)=0$ $\therefore x=0$ **or** $x=6.4$	2		

QUESTION 4

Suggested Solutions	Max Mark	Your Mark	Marker's Comments
(d) (iii)  From symmetry, $OA = a = BC$ Now $OC = 6.4$ $BC = OC - OB$ $= 6.4 - 4$ $=2.4$ $\therefore 0A = a = BC$ $\therefore a = 2.4$	1		

QUESTION 5

Suggested Solutions	Max Mark	Your Mark	Marker's Comments
(a) $let\, u = \log_e x$, then $du = \frac{dx}{x}$ $Now\, if\ x = e^5, \log_e x = 5$ $x = e^2, \log_e x = 2$ $Now\quad I = 2\int_{e^2}^{e^5} \log_e x \frac{dx}{x}$ $= 2\int_2^5 u\, du = \left[u^2\right]_2^5 = 25 - 4 = 21$	3		
(b) Add α and β to get $2\cos^{-1} x = \pi$ $\therefore \cos^{-1} x = \frac{\pi}{2}, giving\ x = 0.$ Using β, $\frac{\pi}{2} - 3\sin^{-1} y = 0,$ $-3\sin^{-1} y = -\frac{\pi}{2}$ $\sin^{-1} y = \frac{\pi}{6} \quad \therefore y = \frac{1}{2}$	3		
(c) (i) L.H.S = $\sin 3A = \sin\frac{\pi}{2} = 1, if\ A = \frac{\pi}{6}$ R.H.S = $3\sin\frac{\pi}{6} - 4\sin^3\frac{\pi}{6} = \frac{3}{2} - \frac{1}{2} = 1$ So L.H.S = R.H.S = 1 if $A = \frac{\pi}{6}$	1		
(ii) Given equation can be written as $3x - 4x^3 = \frac{1}{2}$ $or\ 3\sin A - 4\sin^3 A = \frac{1}{2}, where\ x = \sin A$ $\therefore \sin 3A = \frac{1}{2}\ giving\ smallest\ positive\ value\ of\quad 3A = \frac{\pi}{6}$ $\therefore A = \frac{\pi}{18}$ is the *required answer.*	3		

QUESTION 5

Suggested Solutions	Max Mark	Your Mark	Marker's Comments
(d) Given, $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)\ldots(\delta)$ *Let* $a=(\alpha-\beta), b=(\beta-\gamma), c=(\gamma-\alpha)$ *in the given identity* δ. *Then* $a+b+c=(\alpha-\beta)+(\beta-\gamma)+(\gamma-\alpha)=0$, so *R.H.S of* $\delta=0$. $\therefore a^3+b^3+c^3-3abc.=0$ $\therefore a^3+b^3+c^3=3abc$ $ie(\alpha-\beta)^3+(\beta-\gamma)^3+(\gamma-\alpha)^3=3(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$ $\therefore$*The required* factors are $3(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$	2		

QUESTION 6			
Suggested Solutions	**Max Mark**	**Your Mark**	**Marker's Comments**
(a) (i) Many approaches are possible for example $RHS \quad = \frac{1}{2}\frac{d}{dx}(v^2) = \frac{1}{2}2v\frac{dv}{dx}$ $= \frac{dx}{dt}.\frac{dv}{dx} = \frac{dv}{dt} = acceleration$	1		
(ii) (α) From part (i) $\frac{d}{dx}\left(\frac{v^2}{2}\right) = -n^2x$, (where $n = \frac{2\pi}{T}$, T being the period). Integrate both sides with respect to x to get, $\frac{v^2}{2} = -\frac{n^2x^2}{2} + B$. (B the constant of integration) From data, when $t = 0$, $v = 0$ and $x = a$ $\therefore 0 = -\frac{n^2a^2}{2} + B \Rightarrow B = \frac{n^2a^2}{2} \Rightarrow \frac{v^2}{2} = -\frac{n^2x^2}{2} + \frac{n^2a^2}{2}$. Multiply by 2 to arrive at , $v^2 = n^2(a^2 - x^2)$, $\quad so\ v = \pm n\sqrt{a^2 - x^2}$. But when $t = 0$, $x = a$ (data), so particle must begin by moving to the left. (Note $x = a$ is the maximum positive position of P). $\therefore \quad v = -n\sqrt{a^2 - x^2}, \quad \lvert x \rvert \le a$	2		
(ii) (β) We can re-arrange the result of part (ii) (α) viz, $\frac{dx}{dt} = -n\sqrt{a^2 - x^2}$, $\lvert x \rvert \le a$, to get $-\int \frac{dx}{\sqrt{a^2 - x^2}} = \int n\,dt$ $\therefore \cos^{-1}\left(\frac{x}{a}\right) = nt + C$. (C is the constant of integration). But from data, when $t = 0, x = a$. $\therefore$ C = 0, *as* $\cos^{-1}(1) = 0$ $\therefore \cos^{-1}\left(\frac{x}{a}\right) = nt \Rightarrow$ So $x = a\cos nt$	2		

QUESTION 6

Suggested Solutions	Max Mark	Your Mark	Marker's Comments
(b) Here $x = 2+2\cos\left(\frac{\pi}{390}\right)t$, the standard S.H.M. form $x = a\cos nt$. So $n = \left(\frac{\pi}{390}\right)$ and period $= \frac{2\pi}{n} = 780$ mins So from high tide to next high tide = 780 mins. Hence from high tide to next low tide = 390 mins.	2		
(c) When $t = 65$, $\frac{dx}{dt} = -\frac{\pi}{195}\sin\frac{\pi t}{390} = \frac{-\pi}{195}\sin\frac{\pi 65}{390} = \left(\frac{-\pi}{390}1000\right) mm/\min$ $\approx -8.055...(calc) \approx -8mm/\min$. The minus sign indicates that the water level is falling.	2		
(d) Since the distance between the bridge and the water has to be 7m, clearly x=3 $\therefore 2(1+\cos\frac{\pi t}{390}) = 3,\ giving\ \cos\frac{\pi t}{390} = \frac{1}{2}$ $\therefore \frac{\pi t}{390} = \frac{\pi}{3}.\quad Hence\ t = 130$ The yacht can travel under the bridge 2 hours 10 mins after high tide.	3		

QUESTION 7

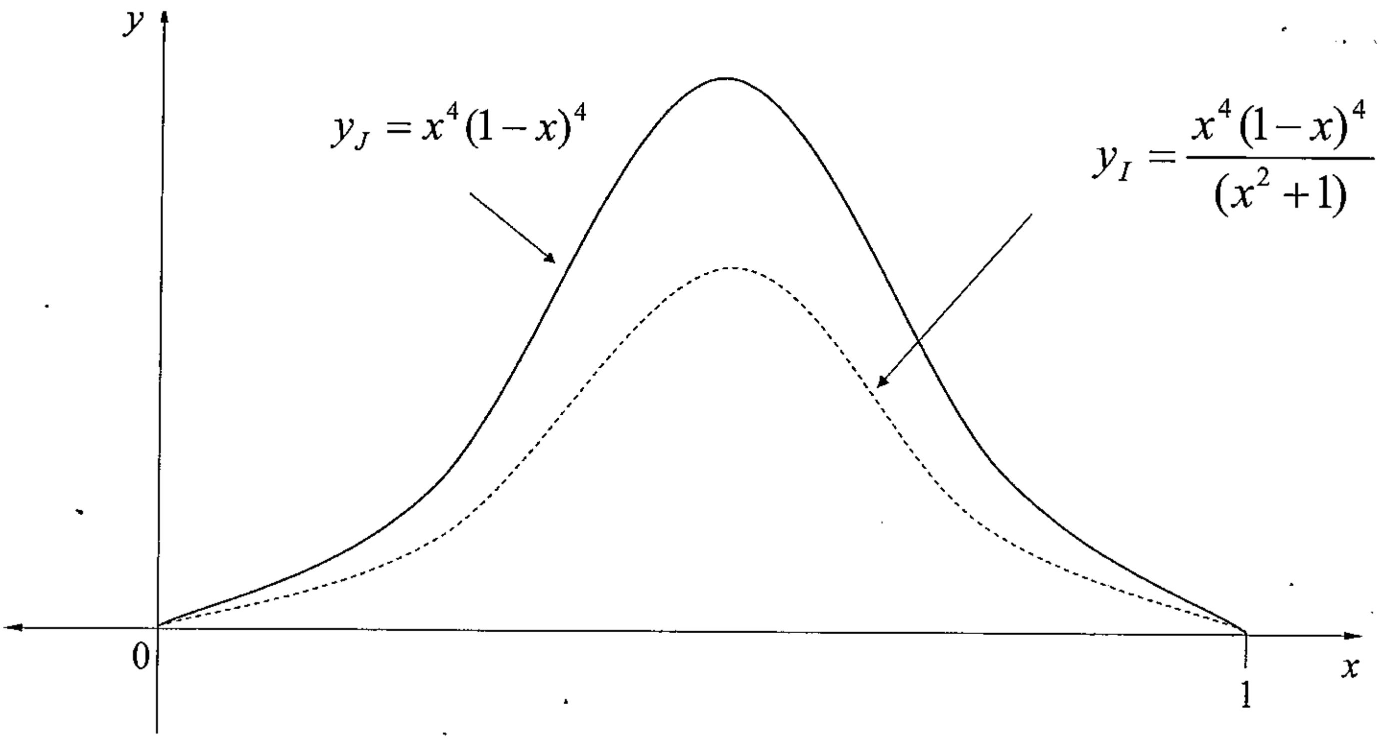
Suggested Solutions	Max Mark	Your Mark	Marker's Comments
(a) $J=\int_0^1 x^4(1-x)^4dx$ $=\int_0^1 (x^8-4x^7+6x^6-4x^5+x^4)dx.........(given)$ $=\left[\frac{x^9}{9}-\frac{x^8}{2}+\frac{6x^7}{7}-\frac{4x^6}{6}+\frac{x^5}{5}\right]_0^1$ $=\frac{1}{9}-\frac{1}{2}+\frac{6}{7}-\frac{2}{3}+\frac{1}{5}$ $=\frac{70-315+540-420+126}{630}$ $=\frac{736-735}{630}=\frac{1}{630}$	3		
(b) First do the long division $x^2+1\,\overline{)\,x^8-4x^7+6x^6-4x^5+x^4}$ quotient $x^6-4x^5+5x^4-4x^2+4$ $\underline{x^8\quad +x^6}$ $-4x^7+5x^6-4x^5+x^4$ $\underline{-4x^7\quad -4x^5}$ $+5x^6\quad +x^4$ $\underline{+5x^6\quad +5x^4}$ $-4x^4$ $\underline{-4x^4-4x^2}$ $4x^2$ $\underline{4x^2+4}$ -4 $\therefore$ E $=x^6-4x^5+5x^4-4x^2+4\quad-\frac{4}{x^2+1}$	3		

QUESTION 7

Suggested Solutions	Max Mark	Your Mark	Marker's Comments
(c) $I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ $= \int_0^1 (x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2})dx ...(\textit{from part b})$ $= \left[\frac{x^7}{7} - \frac{4x^6}{6} + x^5 - \frac{4x^3}{3} + 4x - 4\tan^{-1} x\right]_0^1$ $= \left[\frac{1}{7} - \frac{2}{3} + \frac{5}{5} - \frac{4}{3} + 4\right] - 4\left[\tan^{-1}(1)\right]$ $= \frac{30 - 140 + 210 - 280 + 840}{210} - 4\left(\frac{\pi}{4}\right)$ $= \frac{1080 - 420}{210} - \pi$ $= \frac{660}{210} - \pi$ $= \frac{22}{7} - \pi$	3		
(d) Given that $J > I > \frac{J}{2}$ $\frac{1}{630} > \frac{22}{7} - \pi > \frac{1}{1260}$ *from parts (a) and (c)* (*multiply through by* -1) $-\frac{1}{630} < \pi - \frac{22}{7} < -\frac{1}{1260}$ (add $\frac{22}{7}$ to each expression) $\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}$	2		

QUESTION 7

Suggested Solutions	Max Mark	Your Mark	Marker's Comments
(e) From observation J represents the area under the full curve. I represents the area under the broken curve. So clearly $J > I$.	1		



An Algebraic Proof:

For $0 < x < 1$ we have

$1 < 1 + x^2 < 2$

So $x^4(1-x)^4 > \dfrac{x^4(1-x)^4}{1+x^2} > \dfrac{x^4(1-x)^4}{2}$

$J > I > \dfrac{J}{2}$

Source of Question 7:

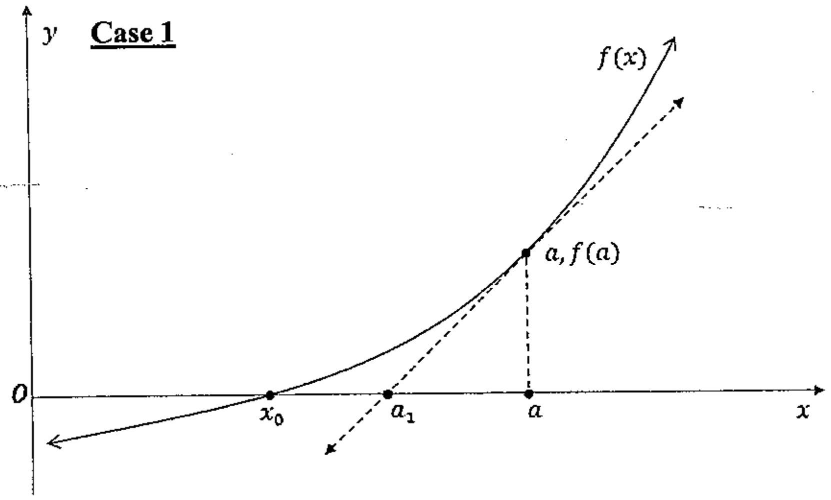
N.S.W. Leaving Certificate Maths 1 Honours Paper 1964, Q9.
Also adapted from a Cambridge Matriculation examination paper, late 1930's.

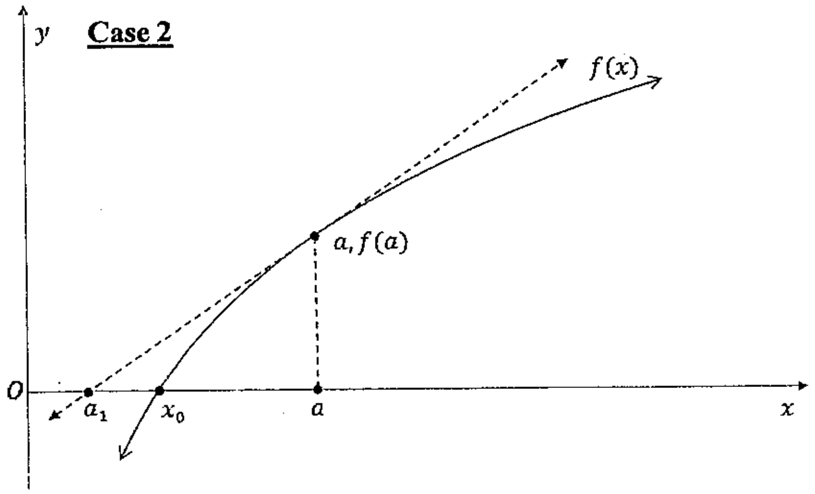
Question (4C)

In the following examples the required root of f(x)=0 is given by x_0.

In cases 1 and 2 below, Newton's method works and a_1 is a better approximation than a.

(i) (2 marks)



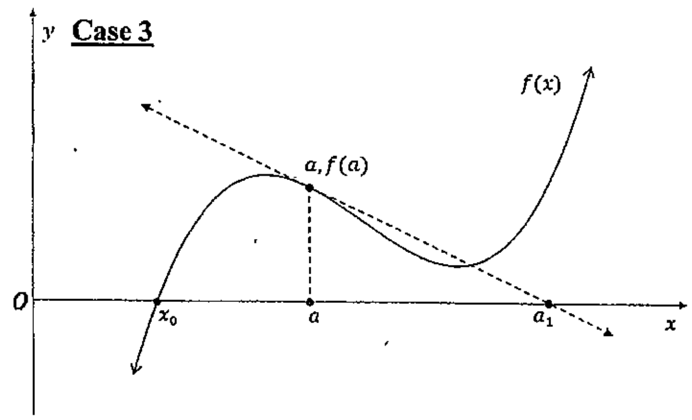


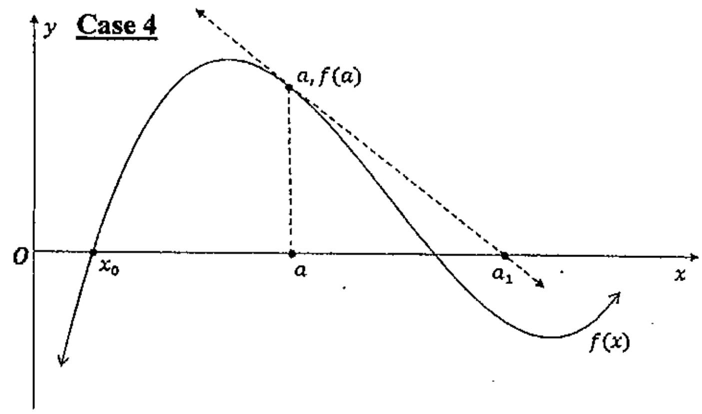
Question (4C)

In cases 3, 4 and 5, Newton's method fails and a_1 is <u>not</u> a better approximation than a.

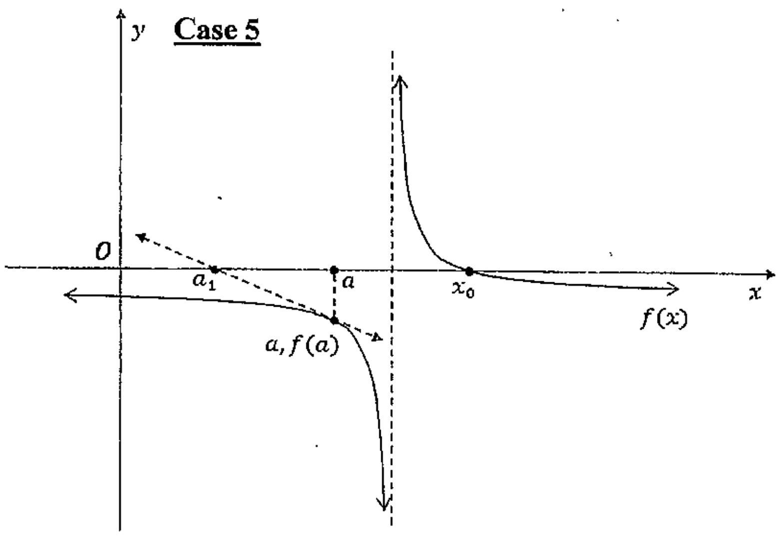
For cases 3 and 4, this is because $f'(x) = 0$ at some point between a and x_o or at $x = a$

(ii) (1 mark)



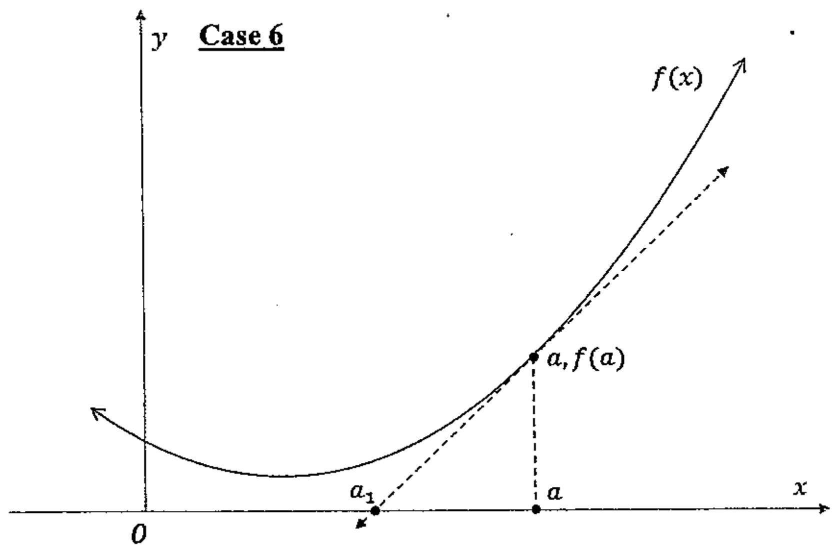


For case 5, $f'(x)$ is undefined at some point between a and x_o



Question (4C)

The original question asks to show a case where Newton's method does not apply "even though $f(x)$ has a root near $x = a$". Case 6 shows a function $f(x)$ that does not have a root at all. Hence, this case is invalid as a solution.



Case 7 below shows a function similar to that in case 1. However, this case illustrates that the first approximation using Newton's method, a_1 is not necessarily closer than the initial approximation a. However, subsequent approximations (a_2, a_3...) will get progressively closer to the required root x_0. This is not necessarily the case in cases 3, 4 and 5 where Newton's method fails.